

Dynamically decoupled three-body interactions with applications to interaction-based quantum metrology

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We propose a stroboscopic method to dynamically decouple the effects of two-body atom-atom interactions for ultracold atoms, and realize a system dominated by elastic three-body interactions. Using this method, we show that it is possible to achieve the optimal scaling behavior predicted for interaction-based quantum metrology with three-body interactions. Specifically, we show that for ultracold atoms quenched in an optical lattice, we can measure the three-body interaction strength with a precision proportional to $\bar{n}^{-5/2}$ using homodyne quadrature interferometry, and $\bar{n}^{-7/4}$ using conventional collapse-and-revival techniques, where \bar{n} is the mean number of atoms per lattice site. Both precision scalings surpass the nonlinear scaling of $\bar{n}^{-3/2}$, the best so far achieved or proposed with a physical system. Our method of achieving a decoupled three-body interacting system may also have applications in the creation of exotic three-body states and phases.

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Introduction.— The ultimate precision of a measurement plays an important role in many areas of physics and technology. Important examples are gravitational wave detection, atomic clocks, and magnetometry [1]. A current goal of metrology research is to improve measurement precision using quantum resources such as squeezing and entanglement [2, 3]. Measurement of a parameter γ using an interferometer with \bar{n} independent particles can achieve a precision limit of $\delta\gamma \propto \bar{n}^{-1/2}$. This is the shot-noise or standard quantum limit (SQL), which is the best precision possible for a classical system. Use of quantum resources can improve this precision to the Heisenberg limit (HL) $\delta\gamma \propto \bar{n}^{-1}$ [4]. Recent experiments with photons, atoms and other systems have achieved sub-shot-noise precision [5, 6].

Recent analyses of the quantum Cramer-Rao bound with nonlinear terms in the Hamiltonian [7–15] have argued that a precision beyond the Heisenberg limit is possible. Multibody (k -body) interactions can give rise to the nonlinearity, where its strength U_k corresponds to the parameter to be estimated. A k -body interaction in the Hamiltonian is predicted to give a scaling of \bar{n}^{-k} using an optimally entangled state and $\bar{n}^{-(k-1/2)}$ even without entanglement, where \bar{n} is the number of probes. This so-called “super-Heisenberg” scaling surpasses the conventional Heisenberg limit for $k \geq 2$, and reduces to the SQL and HL for $k = 1$ (linear case). Scaling of $\bar{n}^{-3/2}$, where \bar{n} is the number of photons, has been experimentally achieved [16] in the detection of atomic magnetization that couples to effective pairwise ($k = 2$) photon-photon interactions. Theory proposals and analysis exist on performing interaction-based metrology with two-body interactions in a number of systems [10, 17–23]. Although there continues to be debate on the correct way to count resources for nonlinear metrologies [2, 24], the potential for either enhanced or new types of measurement exploiting particle-particle interactions in quantum systems deserves further investigation.

In this paper, we propose a method for achieving the optimal precision scaling for an interaction-based quantum metrology exploiting three-body interactions ($k = 3$), within an experimentally realizable physical system. For ultracold atoms in an optical lattice, we show that the elastic three-body interaction strength can be measured with a precision scaling of $\bar{n}^{-5/2}$ using a quadrature method [18] and $\bar{n}^{-7/4}$ using conventional collapse and revival techniques [25–30], where \bar{n} is the average number of atoms per site. These precision limits surpass the interaction-based scaling of $\bar{n}^{-3/2}$, the best possible scaling so far realized [16] or proposed [10, 18] with a physical system. Our analysis and results add to the toolbox of quantum metrology and may find applications in demanding precision measurements.

Ultracold atoms in a shallow optical lattice, when quenched to a deep lattice, exhibit matter-wave collapse and revivals with signatures of multibody interactions [25]. The challenge for exploiting the three-body physics is to effectively turn off or decouple the, typically stronger, influence of the two-body interactions on the dynamics. We propose achieving this via a dynamical decoupling protocol in which a Feshbach resonance is used to switch the sign of U_2 periodically while U_3 is unchanged. This cancels the influence of two-body interactions on the dynamics, decoupling the three-body physics in stroboscopic measurements. This technique for decoupling two- and three-body interactions may also have application to generating novel three-body states with topological characteristics, such as the Pfaffian state [31] and other exotic phases and phenomena [32–35]. We note that one can also modify the multibody interactions by dressing atoms using microwave or radio-frequency radiation [36, 37].

State preparation.— Our system is an ultracold gas of bosons in an optical lattice, which can initially be de-

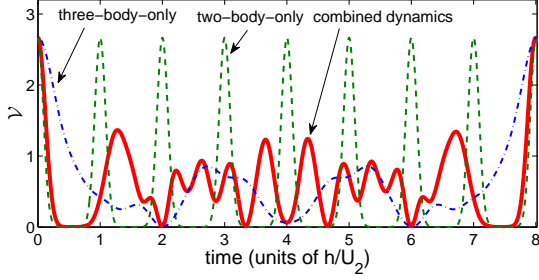


FIG. 1: (color online). Visibility as a function of time showing the effects of two- and three-body interactions in the quench dynamics of ultracold atoms in an optical lattice, where $\bar{n} = 2.67$. The solid line (red) is for the combined U_2 and U_3 dynamics, while the dashed and dot-dashed lines are for U_2 -only and U_3 -only dynamics, respectively. To extract the three-body scaling, we need to isolate the U_3 -only dynamics from the combined time trace.

scribed by the single-band Bose-Hubbard Hamiltonian,

$$H_i = -J_i \sum_{\langle jj' \rangle} (b_j^\dagger b_{j'} + \text{h.c.}) + \frac{U_{2,i}}{2} \sum_j n_j (n_j - 1), \quad (1)$$

where j, j' are indices to lattice sites, $n_j = b_j^\dagger b_j$, J_i is the hopping parameter, $U_{2,i}$ is the initial on-site two-body atom-atom interaction strength, and only nearest-neighbor tunneling is assumed. The Hamiltonian holds for 1D, 2D, and 3D systems. The total number of particles is $N = \sum_j \langle b_j^\dagger b_j \rangle$, where $\bar{n} = N/M$ is the mean occupation per site and M is the number of sites.

We prepare our initial state as a superfluid in a shallow lattice, which in the limit of $U_{2,i}/J_i \rightarrow 0$ approaches a product of coherent states, one at each lattice site. We then suddenly increase (quench) the depth of the optical lattice such that tunneling is suppressed [25]. The effective Hamiltonian for the post-quench dynamics is

$$H_f = \frac{U_2}{2} \sum_j b_j^\dagger b_j^\dagger b_j b_j + \frac{U_3}{6} \sum_j b_j^\dagger b_j^\dagger b_j^\dagger b_j b_j b_j + \mathcal{O}(U_2^3), \quad (2)$$

where U_2 and U_3 are, respectively, the effective two- and three-body interaction strengths in the deep lattice. The effective three-body interaction arises due to collision-induced virtual excitations to higher bands or vibrational levels of the isolated sites [38]. Approximating the bottom of the deep lattice as an isotropic harmonic potential with frequency ω_f , U_3 is attractive and given by [38, 39]

$$U_3 = -c_f U_2^2 / (\hbar \omega_f) + \mathcal{O}(U_2^3), \quad (3)$$

with $c_f = 1.344$. There exist additional higher-body corrections of order $\mathcal{O}(U_2^3)$ whose strengths are much smaller [38], and that are omitted in this article.

The initial superfluid state is not an eigenstate of the deep lattice. Since the initial state is separable and the lattice sites are decoupled after the quench, the state at each site evolves as $|\Psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |n\rangle$, with

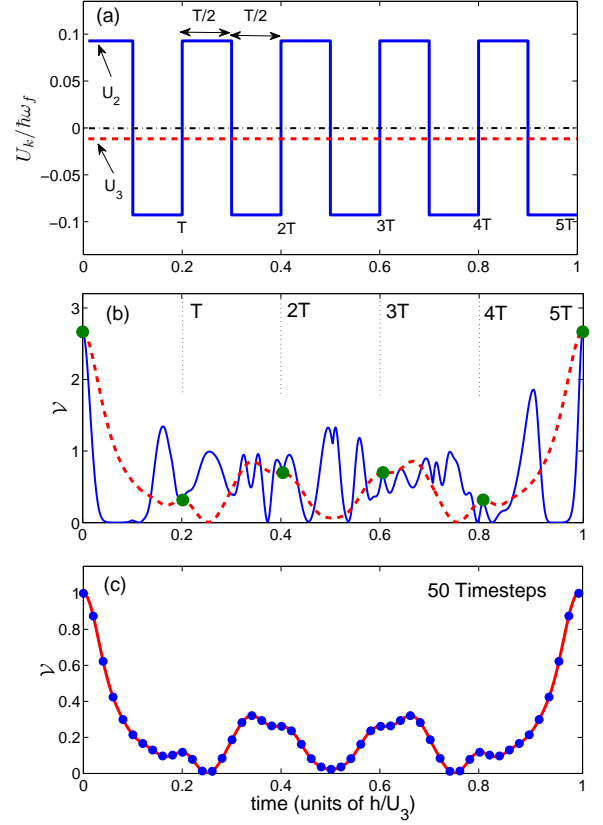


FIG. 2: (color online). A schematic of the dynamical decoupling protocol to effectively turn off the two-body interactions and isolate the dynamics due to the three-body interaction. Panel (a) shows the switching of the sign of U_2 as a function of time, which can be done through the use of a Feshbach resonance. The three-body strength U_3 remains constant and negative. Panel (b) shows the visibility (solid line) as a function of time under the influence of the periodically changing U_2 shown in Panel (a), and $\bar{n} = 2.67$. Also shown as a dashed line is the time evolution due to only U_3 . The two curves intersect after each interval of duration T , indicated by the green circles. Panel (c) shows the intersection points for a time interval T that is 10 times smaller, coinciding even more with the pure three-body time trace. The solid line shows the dynamics for U_3 -only evolution and markers correspond to stroboscopic measurements at multiples of period T .

Fock states $|n\rangle$ containing $n = 0, 1, 2, \dots$ atoms in the lowest vibrational state of a lattice site. The initial amplitudes c_n are given for a coherent state, and the energies are $E_n = U_2 n(n-1)/2 + U_3 n(n-1)(n-2)/6$; $\hbar = h/2\pi$ is the reduced Planck's constant. After a hold-time t in the deep lattice, the lattice is turned off and the atoms expand freely. Absorption imaging of the atomic spatial density yields the quasi-momentum distribution $n_k(t) = (1/M) \sum_{j,j'} e^{ik(j-j')} \rho_{jj'}(t)$, where $\rho_{jj'}(t) = \langle b_j^\dagger b_{j'} \rangle$ and k is the lattice wavevector. A measurement of the normalized observable $\mathcal{V}(t) = n_{k=0}(t)/M$ shows collapse and revival oscillations driven by the two- and three-body interactions [25, 38, 39].

The solid curve in Fig. 1 shows a representative dynamics for the experimentally relevant values of $U_2 = 0.0928\hbar\omega_f$ and $\bar{n} = 2.67$. The effective three-body interaction strength, using Eq. 3, is $U_3 = -0.1247U_2$. We see a complex pattern of oscillations: the faster oscillations are caused by U_2 , modified by a slower envelope due to U_3 . Will *et al.* [25] has observed these predicted visibility oscillations and demonstrated the presence of multi-body interactions by analyzing the oscillation frequencies. The dashed and dot-dashed lines show the simpler behavior that would result from pure U_2 -only and U_3 -only dynamics, respectively. Reference [18] showed that the two-body atom-atom interaction strength U_2 can be extracted from the visibility with a minimal possible uncertainty scaling as $\bar{n}^{-3/2}$, when the measurement is optimized using a quadrature interferometry method, and $\bar{n}^{-3/4}$ without optimization.

Dynamical decoupling protocol.— We propose a protocol similar to dynamical decoupling [40] or spin-echo methods. Our approach is based on the key observation that in Eq. 3 the value of U_3 is independent of the sign of U_2 . This allows one to change U_2 to $-U_2$ using an external magnetic field near a collisional Feshbach resonance [41], without changing the value of U_3 . Specifically, we average out the influence of two-body interactions by alternating between interaction strength set to $|U_2|$, for a time interval $T/2$, and then switching to $-|U_2|$, for the next $T/2$ time interval, thus completing one full time step of duration T . Figure 2(a) shows a schematic of the protocol. As the k -body interaction terms commute, we can write the dynamics in one time-step T in terms of time-ordered unitary evolution operators, giving

$$|\Psi(t + mT)\rangle = \hat{U}_3(U_3, t)\hat{U}_2(|U_2|, t)|\Psi(mT)\rangle \quad (4)$$

for $mT < t < mT + T/2$ where $m = 0, 1, 2, \dots$, and

$$|\Psi(t + mT + T/2)\rangle = \hat{U}_3(U_3, t)\hat{U}_2(-|U_2|, t) \times |\Psi(mT + T/2)\rangle \quad (5)$$

for $mT + T/2 < t < (m + 1)T$. Here $\hat{U}_2(U_2, t) = e^{-iU_2 b^\dagger b^\dagger b b t / (2\hbar)}$ and $\hat{U}_3(U_3, t) = e^{-iU_3 b^\dagger b^\dagger b^\dagger b b b t / (6\hbar)}$. Some algebra shows that at $\tau = T$, $|\Psi(t + T)\rangle = e^{-iU_3 b^\dagger b^\dagger b^\dagger b b b T / (6\hbar)} |\Psi(t)\rangle$; the dynamics due to 2-body interactions cancel exactly at the end of each period T , leaving only the U_3 contribution.

Note that for any other time t , however, both U_2 and U_3 influence the dynamics, giving the complex dynamics depicted in Fig. 2(b). Only for times that are integer multiples of T does the combined evolution yield a visibility that corresponds to the U_3 -only time trace. By making the period T smaller we can obtain a nearly continuous sampling of the U_3 -only dynamics, as depicted in Fig. 2(c). In other words, the protocol gives $|\Psi(mT)\rangle = \hat{U}_3(U_3, T)^m |\Psi(t = 0)\rangle$, for integer m . As an aside, we note that in our method not only two-body but all even-body interactions, such as the effective four-body interaction, are approximately cancelled in the dynamics because their leading-order dependence on U_2 is even.

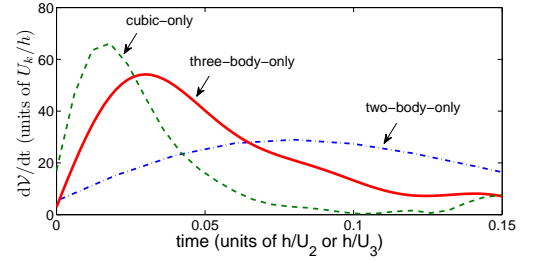


FIG. 3: (color online). The derivative of the visibility \mathcal{V} as a function of time for two-body-only (dot-dashed line), three-body-only (solid), and cubic nonlinearity (dashed) simulations. We use $\bar{n} = 2.67$.

Nonlinear metrology for three-body interactions.— To optimally extract the interaction strength U_3 from the visibility, we need to minimize the fractional uncertainty

$$\delta U_3 / U_3 \propto \frac{\Delta \mathcal{V}}{|d\mathcal{V}/dt|}, \quad (6)$$

obtained by error propagation; $\Delta \mathcal{V}$ is the uncertainty of the visibility, which we obtain from the variance of $n_{k=0}$. Optimal sensitivity is determined by a trade-off between maximizing the derivative and minimizing the uncertainty in the visibility.

Figure 3 shows a comparison of the time derivative of the visibility for two-body-only and three-body-only dynamics. To facilitate comparison, both the x and y axes have been scaled to natural units of time h/U_k , where $k = 2$ or 3 for the two-body-only or three-body-only simulations, respectively. For small times the visibility time trace is steeper for the U_3 -only simulation, in comparison to the U_2 -only simulation. In fact, for larger values of \bar{n} (not shown here), the difference in slope becomes even more pronounced. For U_2 -only dynamics, there exists an analytic expression for the visibility, $\mathcal{V} = \bar{n}e^{2\bar{n}(\cos(U_2 t/\hbar)-1)}$, and its variance, for an initial coherent state. Although no closed-form expression exists for U_3 -only dynamics, for short times and an initial coherent state we can derive a semi-analytic series expansion. Using a combination of analytics and numerics, we find the precision scaling from collapse and revival measurements of \mathcal{V} , given by

$$\delta U_3 / U_3 \propto M^{-1/2} \bar{n}^{-7/4}. \quad (7)$$

The scaling with \bar{n} of $7/4$ is greater than $3/2$, the best so far proposed or achieved exploiting nonlinear interactions [16].

The scaling is valid in the limit $\bar{n} \gg 3$, where $\bar{n}(\bar{n} - 1)(\bar{n} - 2) = \bar{n}^3 - 3\bar{n}^2 + 2\bar{n}$ is to a good approximation \bar{n}^3 and the quadratic correction is negligible. Figure 3 also shows a simulation of the derivative of the visibility when the three-body interaction Hamiltonian is replaced by the cubic nonlinear Hamiltonian $U_3(b^\dagger b)^3/6$. This shows that for $\bar{n} = 2.67$ we have not reached the asymptotic regime of large \bar{n} . We note that $\delta U_3 / U_3$ scales as $M^{-1/2}$, the standard quantum limit in the number of lattice sites.

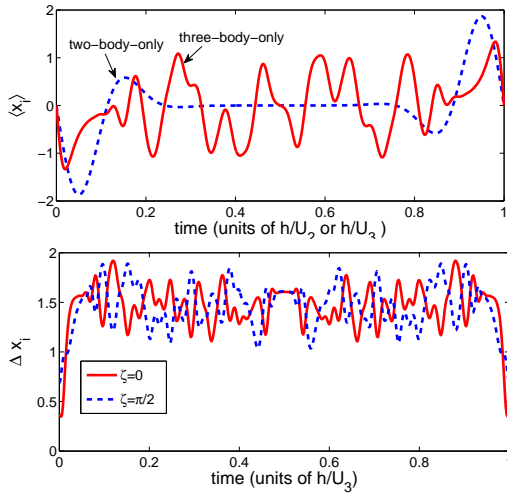


FIG. 4: (color online). Panel (a) quadrature dynamics $\langle x_i \rangle$ as a function of time for two-body-only and three-body-only interactions for $\bar{n} = 4.68$ and $\zeta = 0$. Panel (b) shows the evolution of quadrature fluctuations for three-body-only interactions for $\zeta = 0, \pi/2$ and $\bar{n} = 4.68$.

This is expected as the initial state is site separable and, furthermore, tunneling is turned off after the quench. It is as if we have performed M independent measurements. Consequently, the overall scaling in total particle number $N = \bar{n}M$ is sub-Heisenberg limit. Nevertheless, we find that the improved scaling in \bar{n} promises real improvements in measurement precision.

Measurement theory suggests that for a cubic nonlinearity the best possible precision scaling is \bar{n}^{-3} using entangled states and $\bar{n}^{-5/2}$ using product states. Equation 7 shows that a measurement of visibility in standard collapse and revival experiments does not give the optimal scaling. However, we find that the quadrature interferometry method, details of which are given in Ref. [18], can be used to further improve the scaling behavior. Using that method, we measure the time evolution of the field quadratures, $X_{k=0} = 1/\sqrt{M} \sum_i x_i$, where $x_i = (e^{-i\zeta} b_i + e^{i\zeta} b_i^\dagger)/2$ and ζ is a controllable phase. We then have, $\delta U_3/U_3 \propto M^{-1/2} \Delta x_i / |d\langle x_i \rangle / dt|$ [18], with $\Delta x_i = \sqrt{\langle (x_i - \langle x_i \rangle)^2 \rangle}$, and we can optimize with respect to ζ .

Figure 4(a) shows the dynamics of the quadrature $\langle x_i \rangle$ for two-body-only and three-body-only cases. In comparing the two traces, we see that initially the derivative of the three-body-only case is steeper by a factor of \bar{n} . Figure 4(b) shows the variance Δx_i for three-body-only simulations for different phases ζ . It is smallest for small times and integer multiples of h/U_3 . We choose the phase

variable such that the numerator does not degrade the scaling enhancements gained from the steepness of the time trace. This occurs for $\zeta = \pi/2$, and we find that the best possible scaling is

$$\delta U_3/U_3 \propto M^{-1/2} \bar{n}^{-5/2}. \quad (8)$$

This gives the optimal precision for three-body interactions for an input state that is not entangled. This scaling improves upon the $\bar{n}^{-7/4}$ scaling of Eq. 6.

In principle, we can generalize beyond the use of an initial coherent state, and consider number squeezed initial states (these can be described using the Gutzwiller approximation [42]). Alternative initial states do not affect the dynamical decoupling protocol, and we still obtain stroboscopic evolution under U_3 -only, however, the scaling with \bar{n} will degrade since the measurements are sensitive to phase fluctuations.

Implementation challenges.— Our method promises improved measurement of U_3 even for modest \bar{n} . Recently, Will *et al* [25] obtained $\delta U_3/U_3 = 30\%$ with $N = 2 \times 10^5$ and $\bar{n} = 2.5$ and Ma *et al* [43] obtained $\delta U_3/U_3 = 5\%$. For our proposed method and with $\bar{n} = 4.5$, for example, it is possible in principle to improve the precision by a factor of 6 with the collapse and revivals and 20 with the quadrature method. To achieve the predictions of nonlinear scaling in this paper, experiments need to minimize uncertainties that originate from fluctuations in the total atom number, lattice depth fluctuations from lasers, residual tunneling, and errors in the pulse sequence. Other factors such as three-body recombination losses [41], effective range corrections [39], optical lattice inhomogeneities and the higher-order corrections to the multibody effective interactions [39] may also need to be considered.

Conclusion.— We propose a dynamical decoupling method to average-out the influence of two-body (and higher even-body) interactions in ultracold atom dynamics. Our method for achieving a system dominated by three-body interactions should have a number of applications, including possible realization of novel three-body phases and states. In this paper, we describe how to achieve nonlinear quantum metrology scaling for three-body interactions with an experimentally realizable physical system. We predict a scaling of $\bar{n}^{-5/2}$ using a quadrature method and $\bar{n}^{-7/4}$ in collapse and revivals of momentum distribution. These results are a significant improvement over any scaling so far experimentally achieved or proposed with a physical system exploiting particle-particle interactions.

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